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Non-commutative Geometry Beyond the Standard Model

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Abstract

A natural extension of the standard model within non-commutative geometry is presented. The geometry determines its Higgs sector. This determination is fuzzy, but precise enough to be incompatible with experiment.

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The standard model of electro-weak and strong interactions is a mediocre element of the huge set of Yang-Mills-Higgs theories. Analyzing neighboring theories in this set has essentially two motivations, Lagrange's principle of variation to see whether there is a better theory in the vicinity and an assessment of experimental deviations from the standard model.

Non-commutative geometry allows us — in some cases — to understand the Higgs field as a magnetic field of a Yang-Mills gauge field. There are essentially three approaches to this idea. The approach due to Dubois-Violette, Kerner and Madore [1] is the most restrictive one. It applies to Yang-Mills theories with unbroken parity. The approach due to Coquereaux, Esposito-Farèse and Vaillant [2], on the other hand, is so general that we do not have a model building kit. In the following, we shall stick to Connes' approach [3], that to our taste [4] has the most appealing geometrical motivation. This approach is restricted to a tiny set of Yang-Mills theories with Dirac fermions. That the standard model belongs to this set, is a miracle to us. In order to appreciate it, we look for extensions of the standard model within Connes' frame. These are not easy to find. Left-right symmetric models and grand unified theories do not fit into Connes' frame [5]. No realistic supersymmetric model has been found so far [6].

The mild extension of the standard model, that we would like to discuss here is motivated [7] by the quantum group $SU(2)_j$ with j a cubic root of 1. Non-commutative geometry is the geometry of spaces where points are excluded by an uncertainty relation. The phase space in quantum mechanics is the first example of a non-commutative geometry. Today, the word 'quantum' is so overused that we prefer Madore's terminology. He calls these spaces *fuzzy* and his fuzzy sphere is a most instructive example [8]. According to Connes, the quantum group is to a fuzzy space what the Lie group is to a manifold. So far the quantum group of the standard model is unknown, but the hope is that this quantum group will explain the *fuzzy* mass relation for the Higgs mass [9],

$$m_H^2 \;=\; 3\,{m_t}^2 - m_W^2\left(1+\frac{{g_2}^{-2}}{{g_1}^{-2}-\frac{1}{6}{g_3}^{-2}}\right)\;+\;O\left(\frac{m_\tau^4}{m_t^2}\right) \tag{1}$$

which appears if we want to fit the standard model into Connes' frame. $SU(2)_j$ co-acts on the associative algebra $M_2(\mathbb{C})\oplus M_1(\mathbb{C})\oplus M_3(\mathbb{C})$ which extends mildly the algebra of the standard model, $\mathbb{H}\oplus\mathbb{C}\oplus M_3(\mathbb{C})$.

1 Input

The input of Connes' model building kit is a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ and a non-commutative coupling. $\mathcal{A}$ is an associative involution algebra with unit. Its group of unitaries,

$$G := \{g\in\mathcal{A}\mid g^*g=gg^*=1\}\,, \tag{2}$$

or a subgroup thereof will be the group of gauge transformations. $\mathcal{H}$ is a Hilbert space that carries a faithful representation ρ of $\mathcal{A}$. The Hilbert space is supposed to decompose into four

pieces,

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c, \tag{3}$$

containing the left- and right-handed fermions and anti-fermions,

$$\rho = \begin{pmatrix} \rho_L & 0 & 0 & 0 \\ 0 & \rho_R & 0 & 0 \\ 0 & 0 & \bar\rho_L^c & 0 \\ 0 & 0 & 0 & \bar\rho_R^c \end{pmatrix}. \tag{4}$$

$\mathcal{D}$ is the Dirac operator, an odd, selfadjoint operator on $\mathcal{H}$. $\mathcal{D}$ contains the fermionic mass matrix $\mathcal{M}$,

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathcal{M}} \\ 0 & 0 & \overline{\mathcal{M}^*} & 0 \end{pmatrix}, \tag{5}$$

with respect to the above decomposition of $\mathcal{H}$. The non-commutative coupling z parameterizes invariant scalar products and therefore generalizes the Yang-Mills gauge couplings. z is an even, positive operator on $\mathcal{H}$,

$$z = \begin{pmatrix} z_L & 0 & 0 & 0 \\ 0 & z_R & 0 & 0 \\ 0 & 0 & z_L^c & 0 \\ 0 & 0 & 0 & z_R^c \end{pmatrix}, \tag{6}$$

that commutes with ρ and $\mathcal{D}$.

For the standard model, the spectral triple and the coupling are:

$$\mathcal{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \ni (a, b, c), \tag{7}$$

$\mathbb{H}$ denoting the quaternions,

$$\mathcal{H}_L = \left(\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3\right) \oplus \left(\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}\right), \tag{8}$$

$$\mathcal{H}_R = \left((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3\right) \oplus \left(\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}\right). \tag{9}$$

In each summand, the first factor denotes weak isospin doublets or singlets, the second - N generations, $N = 3$, and the third denotes colour triplets or singlets.
Let us choose the following basis of $\mathcal{H} = \mathbb{C}^{90}$:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L;$$

$$\begin{matrix} u_R, & c_R, & t_R, \\ d_R, & s_R, & b_R, \end{matrix} \quad e_R, \quad \mu_R, \quad \tau_R;$$

$$\begin{pmatrix} u \\ d \end{pmatrix}_L^c, \begin{pmatrix} c \\ s \end{pmatrix}_L^c, \begin{pmatrix} t \\ b \end{pmatrix}_L^c, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L^c, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L^c, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L^c;$$

$$\begin{array}{c} u_R^c, \\ d_R^c, \end{array} \quad \begin{array}{c} c_R^c, \\ s_R^c, \end{array} \quad \begin{array}{c} t_R^c, \\ b_R^c, \end{array} \quad e_R^c, \quad \mu_R^c, \quad \tau_R^c, \tag{10}$$

$$\rho_L(a) = \begin{pmatrix} a\otimes 1_N\otimes 1_3 & 0 \\ 0 & a\otimes 1_N \end{pmatrix}, \qquad \rho_R(b) = \begin{pmatrix} B\otimes 1_N\otimes 1_3 & 0 \\ 0 & \bar{b}1_N \end{pmatrix}, \quad B := \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix},$$

$$\rho_L^c(b,c) = \begin{pmatrix} 1_2\otimes 1_N\otimes c & 0 \\ 0 & \bar{b}1_2\otimes 1_N \end{pmatrix}, \qquad \rho_R^c(b,c) = \begin{pmatrix} 1_2\otimes 1_N\otimes c & 0 \\ 0 & \bar{b}1_N \end{pmatrix}, \tag{11}$$

$$\mathcal{M} = \begin{pmatrix} \begin{pmatrix} M_u\otimes 1_3 & 0 \\ 0 & M_d\otimes 1_3 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 \\ M_e \end{pmatrix} \end{pmatrix}, \tag{12}$$

with

$$M_u := \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad M_d := C_{KM}\begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad M_e := \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \tag{13}$$

All indicated fermion masses are supposed positive and different. The Cabibbo-Kobayashi-Maskawa matrix

$$C_{KM} := \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{14}$$

is supposed non-degenerate in the sense that there is no simultaneous mass and weak interaction eigenstate. The coupling z involves six positive numbers $x,\ y_1,\ y_2,\ y_3,\ \tilde{x},\ \tilde{y}$,

$$z_L = \begin{pmatrix} x/3\,1_2\otimes 1_N\otimes 1_3 & 0 \\ 0 & 1_2\otimes y \end{pmatrix}, \qquad z_R = \begin{pmatrix} x/3\,1_2\otimes 1_N\otimes 1_3 & 0 \\ 0 & y \end{pmatrix},$$

$$z_L^c := \begin{pmatrix} \tilde{x}/3\,1_2\otimes 1_N\otimes 1_3 & 0 \\ 0 & \tilde{y}/3\,1_2\otimes 1_3 \end{pmatrix}, \qquad z_R^c = \begin{pmatrix} \tilde{x}/3\,1_2\otimes 1_N\otimes 1_3 & 0 \\ 0 & \tilde{y}/3\,1_3 \end{pmatrix}, \tag{15}$$

with

$$y := \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix}. \tag{16}$$

Note that on the level of algebra representation, we have an asymmetry between particles and anti-particles, the former are subject to weak, the latter to strong interactions. This asymmetry is lifted later at the level of the Lie algebra representation $\tilde{\rho}$.

The proposed extension of the standard model is mild, we extend the quaternions to arbitrary complex 2×2 matrices, $a \in M_2(\mathbb{C})$,

$$\mathcal{A} = M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_3(\mathbb{C}). \tag{17}$$

All other input items are unchanged. Nevertheless, compared to the standard model, the calculations will turn out to be quite different and longer.

2 Turning the crank

We shall organize our calculations according to the theorem of [10] with invariant scalar product $\mathrm{Re}\,\mathrm{tr}[\rho(\cdot)^*\rho(\cdot)\,z]$. Our first task is to compute the 1-forms. ρ^c being vectorlike does not produce 1-forms and momentarily we may restrict ourselves to $\mathcal{H}_L \oplus \mathcal{H}_R$. A general 1-form is a sum of terms

$$\pi((a_0,b_0)\delta(a_1,b_1)) = -i\begin{pmatrix} 0 & \rho_L(a_0)\,(\mathcal{M}\rho_R(b_1)-\rho_L(a_1)\mathcal{M}) \\ \rho_R(b_0)\,(\mathcal{M}^*\rho_L(a_1)-\rho_R(b_1)\mathcal{M}^*) & 0 \end{pmatrix} \tag{18}$$

and the vector space of 1-forms is

$$\Omega^1_{\mathcal{D}}\mathcal{A} = \left\{ i\begin{pmatrix} 0 & \rho_L(h)\mathcal{M} \\ \mathcal{M}^*\rho_L(\tilde{h}^*) & 0 \end{pmatrix},\ h,\tilde{h}\in M_2(\mathbb{C})\right\}. \tag{19}$$

Our basic variable, the 'Higgs', is an anti-Hermitian 1-form

$$H = i\begin{pmatrix} 0 & \rho_L(h)\mathcal{M} \\ \mathcal{M}^*\rho_L(h^*) & 0 \end{pmatrix},\quad h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \in M_2(\mathbb{C}). \tag{20}$$

It is parameterized by *two* isospin doublets

$$h_1 = \begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix},\quad h_2 = \begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix}. \tag{21}$$

Our next task is to compute the 2-forms. The junk in degree two is:

$$J^2 = \left\{ i\begin{pmatrix} j\otimes\Delta & 0 \\ 0 & 0 \end{pmatrix},\quad j\in M_2(\mathbb{C})\right\} \tag{22}$$

with

$$\Delta := \frac{1}{2}\begin{pmatrix} (M_uM_u^* - M_dM_d^*)\otimes 1_3 & 0 \\ 0 & -M_eM_e^* \end{pmatrix}. \tag{23}$$

With respect to the scalar product the 2-forms are written as

$$\Omega^2_{\mathcal{D}}\mathcal{A} = \pi(\Omega^2\mathcal{A})/J^2 = \left\{ \begin{pmatrix} \tilde{c}\otimes\Sigma' & 0 \\ 0 & \mathcal{M}^*\rho_L(c)\mathcal{M} \end{pmatrix},\quad \tilde{c},c\in M_2(\mathbb{C})\right\} \tag{24}$$

with

$$\Sigma' = \Sigma - \eta\frac{\mathrm{tr}(\Sigma\Delta z_\ell)}{\mathrm{tr}(\Delta^2 z_\ell)}\Delta, \tag{25}$$

$$z_\ell = \begin{pmatrix} (x/3)1_N\otimes 1_3 & 0 \\ 0 & y \end{pmatrix},\qquad \Sigma := \frac{1}{2}\begin{pmatrix} (M_uM_u^* + M_dM_d^*)\otimes 1_3 & 0 \\ 0 & M_eM_e^* \end{pmatrix}; \tag{26}$$

$$\eta = 1 \quad \text{for the case} \quad \tilde{c},c\in M_2(\mathbb{C}), \tag{27}$$

$$\eta = 0 \quad \text{for the case} \quad \tilde{c},c\in \mathbb{H} \qquad \text{(the standard model)}. \tag{28}$$

For the differential $\delta : \Omega^1_{\mathcal{D}}\mathcal{A} \longrightarrow \Omega^2_{\mathcal{D}}\mathcal{A}$ we have:

$$i\begin{pmatrix} 0 & \rho_L(h)\mathcal{M} \\ \mathcal{M}^*\rho_L(\tilde{h}^*) & 0 \end{pmatrix} \longmapsto \begin{pmatrix} (h+\tilde{h}^*)\otimes\Sigma' & 0 \\ 0 & \mathcal{M}^*\rho_L(h+\tilde{h}^*)\mathcal{M} \end{pmatrix}. \tag{29}$$

Now we can compute the curvature

$$C := \delta H + H^2 = \begin{pmatrix} (h+h^*-hh^*)\otimes\Sigma' & 0 \\ 0 & M^*\rho_L(h+h^*-h^*h)M \end{pmatrix}, \tag{30}$$

where

$$\Phi := H - i\begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^* & 0 \end{pmatrix} =: i\begin{pmatrix} 0 & \rho_L(\varphi)\mathcal{M} \\ \mathcal{M}^*\rho_L(\varphi^*) & 0 \end{pmatrix},\ \varphi = h-1, \qquad \varphi = (\varphi_1,\varphi_2),$$
$$\varphi, h \in M_2(\mathbb{C}), \tag{31}$$

is the homogeneous scalar variable.

In order to compute the Higgs potential [10] we must return to $\mathbb{C}^{90}$,

$$V := \mathrm{Re}\,\mathrm{tr}\left[(C-\alpha C)^*(C-\alpha C)\,z\right]. \tag{32}$$

We need to know the linear map

$$\alpha : \Omega^2_{\mathcal{D}}\mathcal{A} \longrightarrow \rho(\mathcal{A}) + J^2 \tag{33}$$

which is determined by the two equations

$$\mathrm{Re}\,\mathrm{tr}\left[R^*(C-\alpha C)\,z\right] = 0 \qquad \text{for all } R \in \rho(\mathcal{A}), \tag{34}$$
$$\mathrm{Re}\,\mathrm{tr}\left[K^*\alpha C\,z\right] = 0 \qquad \text{for all } K \in J^2. \tag{35}$$

The solution of $(34, 35)$ is given by

$$\alpha C = \begin{pmatrix} \rho_L(a) & 0 & 0 & 0 \\ 0 & \rho_R(b) & 0 & 0 \\ 0 & 0 & \bar{\rho}^c_L(b,0) & 0 \\ 0 & 0 & 0 & \bar{\rho}^c_R(b,0) \end{pmatrix} + i\begin{pmatrix} k\otimes\Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

with

$$a = c\,\frac{\mathrm{tr}(\Sigma' z_\ell)}{\mathrm{tr}z_\ell - \eta\left[\mathrm{tr}(\Delta z_\ell)\right]^2/\mathrm{tr}(\Delta^2 z_\ell)}, \quad ik = -c\,\frac{\mathrm{tr}(\Sigma' z_\ell)}{\mathrm{tr}z_\ell - \eta\left[\mathrm{tr}(\Delta z_\ell)\right]^2/\mathrm{tr}(\Delta^2 z_\ell)}\,\frac{\mathrm{tr}(\Delta z_\ell)}{\mathrm{tr}(\Delta^2 z_\ell)}\,\eta,$$

$$b = \frac{c_{11}\mathrm{tr}(M_u^* M_u)x + c_{22}\mathrm{tr}(M_d^* M_d)x + c_{22}\mathrm{tr}(M_e^* M_e y)}{2Nx + \mathrm{tr}y + 3\tilde{y}}, \quad c := h + h^* - hh^* = 1 - \varphi\varphi^*.$$

The Higgs potential is:

$$V = \mathrm{tr}(c^2)\left(\mathrm{tr}\left(\Sigma'^2 z_\ell\right) - \frac{\left[\mathrm{tr}(\Sigma' z_\ell)\right]^2}{\mathrm{tr}z_\ell - \eta\left[\mathrm{tr}(\Delta z_\ell)\right]^2/\mathrm{tr}(\Delta^2 z_\ell)}\right)$$

$$+\left[|c_{11}|^2\mathrm{tr}(M_u^*M_u)^2+2|c_{12}|^2\mathrm{tr}(M_dM_d^*M_u^*M_u)+|c_{22}|^2\mathrm{tr}((M_d^*M_d)^2)\right]x$$

$$+|c_{22}|^2\mathrm{tr}((M_e^*M_e)^2y)-\frac{\left[c_{11}\mathrm{tr}(M_u^*M_u)x+c_{22}\mathrm{tr}(M_d^*M_d)x+c_{22}\mathrm{tr}(M_e^*M_ey)\right]^2}{2Nx+\mathrm{tr}y+3\tilde{y}}, \quad (37)$$

where

$$c_{11}=1-\varphi_1^*\varphi_1, \quad c_{22}=1-\varphi_2^*\varphi_2, \quad c_{12}=-\varphi_1^*\varphi_2, \quad c_{21}=-\varphi_2^*\varphi_1. \quad (38)$$

Let us reparameterize the scalars:

$$h:=\begin{pmatrix} H+ih_Z & -h'^* \\ h & H'-ih'_Z \end{pmatrix}, \quad H,H',h_Z,h'_Z\in\mathbb{R}, \quad h,h'\in\mathbb{C}. \quad (39)$$

From $(37,39)$ we get:

$$\begin{aligned} V \;&=\; 4B_1H^2+4B_2H'^2+4B_3HH'+B_4\left(|h|^2+|h'|^2-\left(h^*h'+h'^*h\right)\right) \\ &\quad\; + \text{terms of order 3 and 4}, \end{aligned} \quad (40)$$

where

$$B_1=A_1+A_2, \quad B_2=A_1+A_3, \quad B_3=A_5, \quad B_4=2A_1+A_4; \quad (41)$$

$$A_1 \;=\; \mathrm{tr}\left(\Sigma'^2z_\ell\right)-\frac{\left[\mathrm{tr}(\Sigma'z_\ell)\right]^2}{\mathrm{tr}z_\ell-\eta\left[\mathrm{tr}(\Delta z_\ell)\right]^2/\mathrm{tr}(\Delta^2z_\ell)}, \quad (42)$$

$$A_2 \;=\; x\,\mathrm{tr}(M_u^*M_u)^2-\frac{L_1}{2Nx+\mathrm{tr}y+3\tilde{y}}, \quad (43)$$

$$A_3 \;=\; x\,\mathrm{tr}(M_d^*M_d)^2+\mathrm{tr}((M_e^*M_e)^2y)-\frac{L_2}{2Nx+\mathrm{tr}y+3\tilde{y}}, \quad (44)$$

$$A_4 \;=\; 2x\,\mathrm{tr}(M_u^*M_uM_dM_d^*), \quad (45)$$

$$A_5 \;=\; \frac{2L_3}{2Nx+\mathrm{tr}y+3\tilde{y}}; \quad (46)$$

$$L_1 \;=\; [x\mathrm{tr}(M_u^*M_u)]^2, \quad (47)$$

$$L_2 \;=\; [x\mathrm{tr}(M_d^*M_d)]^2+[x\mathrm{tr}(M_e^*M_ey)]^2+2x\mathrm{tr}(M_d^*M_d)\mathrm{tr}(M_e^*M_ey), \quad (48)$$

$$L_3 \;=\; 2\left(x^2\mathrm{tr}(M_u^*M_u)\mathrm{tr}(M_d^*M_d)+x\mathrm{tr}(M_u^*M_u)\mathrm{tr}(M_e^*M_ey)\right). \quad (49)$$

To get the physical variables, we must diagonalize simultaneously the mass matrix (40) and the kinetic term in the Klein-Gordon action. The latter has the form:

$$\mathrm{tr}\,(\,\mathrm{d}\Phi^**\,\mathrm{d}\Phi\,z)=\tfrac{1}{2}\,c_1\{(\partial H)^2+(\partial h_Z)^2+|\partial h|^2\}+\tfrac{1}{2}\,c_2\{(\partial H')^2+(\partial h'_Z)^2+|\partial h'|^2\}, \quad (50)$$

where

$$c_1 = 4x\,\mathrm{tr}(M_u^* M_u), \quad c_2 = 4(x\mathrm{tr}(M_d^* M_d) + \mathrm{tr}(M_e^* M_e y)). \tag{51}$$

We obtain:

$$V = \frac{1}{2}\,m_{H_0}^2 H_0^2 \;+\; \frac{1}{2}\,m_{H_0'}^2 H_0'^2 \;+\; \frac{1}{2}\,m_{H^\pm}^2 |H^\pm|^2 \;+\; \text{terms of order 3 and 4}, \tag{52}$$

where

$$\begin{aligned}
H_0 &= \cos\theta_0\sqrt{c_1}H - \sin\theta_0\sqrt{c_2}H', \\
H_0' &= \sin\theta_0\sqrt{c_1}H + \cos\theta_0\sqrt{c_2}H', \\
H^\pm &= \cos\theta_1\sqrt{c_1}h - \sin\theta_1\sqrt{c_2}h', \\
h_W &= \sin\theta_1\sqrt{c_1}h + \cos\theta_1\sqrt{c_2}h',
\end{aligned} \tag{53}$$

with

$$\tan 2\theta_0 = \frac{2c}{b-a}, \quad \tan 2\theta_1 = \frac{2c'}{c_2' - c_1'}, \tag{54}$$

$$a = \frac{4B_1}{c_1}, \quad b = \frac{4B_2}{c_2}, \quad c = -\frac{2B_3}{\sqrt{c_1 c_2}}; \tag{55}$$

$$c_1' = \frac{B_4}{c_1}, \quad c_2' = \frac{B_4}{c_2}, \quad c' = \sqrt{c_1' c_2'}, \tag{56}$$

$\theta_0,\ \theta_1$ the Cabibbo like angles.

The masses of the Higgs particles are given by

$$\begin{array}{lll}
m_{H_0}^2 = a + b + \sqrt{4c^2 + (b-a)^2}, & m_{H_0'}^2 = a + b - \sqrt{4c^2 + (b-a)^2}, & m_{H^\pm}^2 = 2(c_1' + c_2'), \\
\qquad m_{h_Z} = 0, & \qquad m_{h_Z'} = 0, & m_{h_W} = 0.
\end{array} \tag{57}$$

The masses of the gauge bosons (see Table 1 below) are found in the covariant Klein-Gordon Lagrangian,

$$\mathrm{tr}\,(\,\mathrm{D}\Phi^* * \mathrm{D}\Phi\, z) \ \text{ with } \ \mathrm{D}\Phi = \mathrm{d}\Phi + [\rho(A)\Phi - \Phi\rho(A)] \tag{58}$$

the covariant derivative of Φ. The normalisation of the gauge bosons is fixed by their kinetic term in the Yang-Mills Lagrangian $\mathrm{tr}\,(\rho(F) * \rho(F)\, z)$, with $\rho(F) := \mathrm{d}\rho(A) + \rho(A)^2 \in \Omega^2(M, \rho(\mathfrak{g}))$. $\mathfrak{g}$ is the Lie algebra of the group of unitaries,

$$\mathfrak{g} = u(2) \oplus u(1) \oplus u(3) = su(3) \oplus su(2) \oplus u(1)^3. \tag{59}$$

This normalisation introduces the gauge couplings g_i. The Lie algebra $\mathfrak{g}$ being a sum of five ideals one might expect five gauge couplings. However, the basic object in non-commutative geometry is the associative algebra $\mathcal{A}$ which is only a sum of three ideals,

$$g_3^{-2} = \frac{4}{3}N\tilde{x}, \tag{60}$$

$$g_2^{-2} = Nx + \mathrm{tr}y, \tag{61}$$

$$g_1^{-2} = Nx + \frac{2}{9}N\,\tilde{x} + \frac{1}{2}\,\mathrm{tr}y + \frac{3}{2}\mathrm{tr}\tilde{y}. \tag{62}$$

For g_1 we have chosen the gauge coupling of the standard hypercharge.

3 Output

While the non-commutative version of the standard model has one isospin doublet of scalars, the present extension has two. There are now four neutral scalars, two are massless, the Goldstone bosons h_Z and h'_Z and two are massive, the 'physical' Higgs bosons H_0 and H'_0. There are two charged scalars, the massless Goldstone boson h_W and the massive Higgs boson $H^\pm$. The neutral and charged Higgses mix with Cabibbo like angles θ_0 and θ_1. In the neutral sector, the masses and the angle are fuzzy already in the approximation $m_b \ll m_t$, $m_\tau = m_c = \cdots = 0$. Note that in this approximation $A_1 = 0$.

$$m_{H_0}^2 = \left[2m_t^2 + \frac{8}{(5+z)^2} m_b^2\right]\left(1 - \frac{1}{6+z}\right), \tag{63}$$

$$m_{H'_0}^2 = \left[2m_b^2 - \frac{8}{(5+z)^2} m_b^2\right]\left(1 - \frac{1}{6+z}\right), \tag{64}$$

$$\tan 2\theta_0 = \frac{4}{5+z} \frac{m_t m_b}{m_t^2 - m_b^2}, \tag{65}$$

where we have put $z := \mathrm{tr} y/x + 3\tilde{y}/x > 0$ and therefore

$$6 + z = 2(g_1^{-2} - \tfrac{1}{6} g_3^{-2})/x, \tag{66}$$

Let us recall the experimental values of pole masses and gauge couplings at energies of the Z: $m_b = 4.3 \pm 0.2$ GeV, $m_t = 180 \pm 12$ GeV, $g_1 = 0.3575 \pm 0.0001$, $g_2 = 0.6507 \pm 0.0007$, $g_3 = 1.207 \pm 0.026$. Consequently x ranges from 0 to $g_2^{-2}/3$ and z ranges from 13.5 to ∞ and

$$1.38\, m_t \,<\, m_{H_0} \,<\, 1.41\, m_t, \tag{67}$$
$$1.38\, m_b \,<\, m_{H'_0} \,<\, 1.41\, m_b, \tag{68}$$
$$0 \,<\, \sin\theta_0 \,<\, 0.002. \tag{69}$$

Phenomenologically, the light Higgs H'_0 is a disaster in any case. The mass of the charged Higgs $H^\pm$ and their Cabibbo like angle θ_1 are sharp in the above approximation,

$$m_{H^\pm} = m_t + \frac{1}{2} m_b \frac{m_b}{m_t}, \tag{70}$$

$$\tan 2\theta_1 = 2 \frac{m_t m_b}{m_t^2 - m_b^2}, \tag{71}$$

$$\sin\theta_1 = 0.02. \tag{72}$$

Taking into account the τ mass, however will also render these equations fuzzy.

Table 1: Properties of neutral gauge bosons

$\cdot$	γ	γ'	X	Z'	Z
g	$g_2 \sin\theta_w$	e'	g_X	g_2	$g_2 \cos\theta_w$
m	0	0	m_X	m_W	$m_W/\cos\theta_w$
u_L	2/3	$1/2-r$	1/2	1/2	$1/2 - 1/6\,\tan^2\theta_w$
d_L	-1/3	$-1/2-r$	-1/2	1/2	$-1/2 - 1/6\,\tan^2\theta_w$
ν_L	0	0	$1/2 - 1/2\,\tan^2\theta_w$	1/2	$1/2 - 1/2\,\tan^2\theta_w$
e_L	-1	-1	$-1/2 - 1/2\,\tan^2\theta_w$	1/2	$1/2 - 1/2\,\tan^2\theta_w$
u_R	2/3	$1/2-r$	$-1/2\,\tan^2\theta_w$	0	$-2/3\,\tan^2\theta_w$
d_R	-1/3	$-1/2-r$	$1/2\,\tan^2\theta_w$	0	$1/3\,\tan^2\theta_w$
e_R	-1	-1	$\tan^2\theta_w$	0	$\tan^2\theta_w$

Concerning the gauge bosons only the chargeless sector is modified with respect to the standard model. To start, we have four neutral bosons, two massless ones, γ, the genuine photon, and γ', and two massive ones, X and Z'. In the standard model, the Z' is absent and an algebraic condition ('unimodularity') added *ad hoc* reduces the group of gauge transformations G by one $U(1)$ factor and eliminates a linear combination of γ' and X leaving only the photon and the genuine Z. In the standard model, the unimodularity is equivalent to the condition of vanishing gauge anomaly [11],

$$\mathrm{tr}[\chi\epsilon\tilde\rho(X)^3] = 0, \quad \text{for all } X \in \mathfrak{g}, \tag{73}$$

where $\mathfrak{g} := \{X \in \mathcal{A} \mid X^* + X = 0\}$ is the Lie algebra of the group of unitaries G and

$$\tilde\rho(X) := \rho(X) + J\rho(X)J^{-1} \tag{74}$$

is the Lie algebra representation that restores invariance under charge conjugation. χ is the chirality operator, ϵ the projector on the particles and J the charge conjugation. With respect to the decomposition $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c$ they read:

$$\chi = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} C, \tag{75}$$

where C is the charge conjugation of Dirac spinors. Table 1 recollects the physical properties of the neutral gauge bosons, mass, gauge coupling and fermion charges. We have used the following abbreviations:

$$m_X \ := \ (g_2/g_X)m_W \ \approx \ m_Z, \tag{76}$$

$$g_X^2 \;:=\; g_2^2\,\frac{g_2^2[1-g_1^2/(6g_3^2)]}{g_1^2+g_2^2[1-g_1^2/(6g_3^2)]} \;\approx\; g_2^2\cos^2\theta_w, \tag{77}$$

$$e'^2 \;:=\; e^2\, g_1^2/(6g_3^2)\, \cos^2\theta_w\, \left[1-g_1^2\cos^2\theta_w/(6g_3^2)\right]^{-1} \;\approx\; 0.011\, e^2, \tag{78}$$

$$r \;:=\; (g_1^{-2}+g_2^{-2})/g_3^{-2}. \tag{79}$$

These approximations are good at the percent level, $g_1^2/(6g_3^2) = 0.015$.

We note that the gauge coupling of the Z' is sharp whereas in the standard model all gauge couplings are fuzzy. This sharpness comes from the fact that $M_2(\mathbb{C})$ is simple while $U(2)$, its group of unitaries, is not. Phenomenologically, the Z' with its low mass *and* high couplings to fermions is a disaster. On top, the Z' has a gauge anomaly. We are tempted to eliminate it with a second unimodularity condition. Then however, its Goldstone boson remains, another disaster.

Recent experimental evidence for deviations from the standard model in the hadronic sector has motivated an additional neutral gauge boson Z' with a mass around 1 TeV [12]. Clearly this Z' cannot be accommodated in the model discussed here. There remains only one other possibility adding a Z' to Connes' version of the standard model, $\mathcal{H}\oplus\mathbb{C}\oplus M_3(\mathbb{C})$, namely to increase his algebra to $\mathcal{H}\oplus\mathbb{C}\oplus M_3(\mathbb{C})\oplus\mathbb{C}$. Then again, it seems impossible to have a Z' mass above the top mass.

Our conclusion is that within the frame of non-commutative geometry, it is not easy to fiddle around the standard model.

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Note added: Alain Connes has just published a theorem [13] which turns his geometrical motivation mentioned in the introduction into deep mathematics. This theorem also unifies the standard model with general relativity. One of the outcomes of this unification is precisely *the* unimodularity, that reduces the X and γ' to Z and that had remained unexplained so far. In this spirit, a second unimodularity to eliminate the Z' from the $M_3(\mathbb{C})\oplus M_2(\mathbb{C})\oplus M_1(\mathbb{C})$ model is not available.

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